Fractions and Rational Numbers

- A **rational number** is a real number which can be represented as a quotient of two integers.
- Rational numbers are exactly those real numbers having decimal expansions which are **periodic** or **terminating**.

Definition. A **rational number** is a real number which can be written as $\frac{a}{b}$, where a and b are integers and $b \neq 0$. A real number which is not rational is **irrational**.

Example. If p is prime, then $\sqrt{p}$ is irrational.

To prove this, suppose to the contrary that $\sqrt{p}$ is rational. Write $\sqrt{p} = \frac{a}{b}$, where a and b are integers and $b \neq 0$. I may assume that $(a, b) = 1$ — if not, divide out any common factors.

Now

$$b\sqrt{p} = a \quad \text{so} \quad b^2 p = a^2.$$

Since $p \mid a^2$ and p is prime, $p \mid a$. Write $a = pc$. Then

$$b^2 p = p^2 c^2, \quad \text{so} \quad b^2 = pc^2.$$

Now $p \mid b^2$, so $p \mid b$. Thus, p is a common factor of a and b contradicting my assumption that $(a, b) = 1$. It follows that $\sqrt{p}$ is irrational.

More generally, if $a_0, \ldots, a_{n-1}$ are integers, the roots of

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

are either integers or irrational. □

If b is an integer such that $b > 1$, and a is a real number between 0 and 1 (inclusive), then a can be written uniquely in the form

$$a = \sum_{i=1}^{\infty} a_i \cdot \frac{1}{b^i}.$$

This is called the **base b expansion of a**. Rather than proving this fact, I'll merely recall the standard algorithm for computing such an expansion: Subtract from a as many $\frac{1}{b}$'s as possible, subtract as many $\frac{1}{b^2}$'s from what's left, and so on.

Here is a recursive procedure which generates base b expansions:

$$x_0 = a$$

$$a_i = [b \cdot x_{i-1}], \quad x_i = b \cdot x_{i-1} - [b \cdot x_{i-1}] \quad \text{for} \quad i \geq 1.$$

To see why this corresponds to the standard algorithm, note that at the first stage I'm trying to find $k \geq 0$ such that

$$a - \frac{k}{b} \geq 0 \quad \text{and} \quad a - \frac{k+1}{b} < 0.$$

These equations are equivalent to

$$ba - k \geq 0 \quad \text{and} \quad ba - (k+1) < 0, \quad \text{or}$$

$$ba \geq k \quad \text{and} \quad ba < k+1.$$

That is, $k = [ba]$, and a corresponds to x_i.

It's convenient to arrange the computations in a table, as shown below.

Example. Find 0.4 in base 7.

I fill in the rows from left to right. Starting with an x, multiply by $b = 7$ to fill in the third column. Take the greatest integer of the result to fill in the a-column of the next row. Subtract the a-value from the last bx-value to get the next x, and continue. You can check that this is the algorithm described above.

a	x	bx
–	0.4	2.8
2	0.8	5.6
5	0.6	4.2
4	0.2	1.4
1	0.4	2.8

The expansion clearly repeats after this, since I'm getting 0.4 for x again. Thus,

$$0.4 = (0.\overline{2541})_7. \quad \square$$

Definition. The decimal expansion $x = .a_1a_2\ldots$ **terminates** if there is a number $N > 0$ such that $a_n = 0$ for $n \geq N$.

In this case,

$$x = \frac{a_1 \cdot 10^{N-1} + a_2 \cdot 10^{N-2} + \cdots + a_N}{10^N}.$$

Hence, x is rational.

A decimal expansion $x = .a_1a_2\ldots$ is **periodic** with period k if there is a positive integer N such that $a_n = a_{n+k}$ for all $n \geq N$.

Periodic expansions also represent rational numbers. Again, I'll give an example rather than writing out the unenlightening proof.

The converse is also true: Rational numbers have decimal expansions which are either periodic or terminating.

Example. Express $0.\overline{473}$ as a rational number in lowest terms.

Since the number has period 3, I multiply both sides by 10^3:

$$x = 0.\overline{473} = 0.473473\ldots,$$

$$1000x = 473.473473\ldots.$$

Next, subtract the first equation from the second:

$$999x = 473,$$

$$x = \frac{473}{999}. \quad \square$$

Example. Express $(0.\overline{473})_8$ as a (decimal) rational number in lowest terms.

Since the number has period 3, I multiply both sides by $8^3 = 512$:

$$x = (0.\overline{473})_8 = (0.473473\ldots)_8,$$

$$(512)_{10} \cdot x = (473.473473\ldots)_8.$$

Next, subtract the first equation from the second, being careful about the bases:

$$(511)_{10}x = (473)_8 = (315)_{10},$$

$$x = \frac{315}{511} = \frac{45}{73}. \quad \square$$

Example. Express $(0.24\overline{122})_{10}$ as a rational number in lowest terms.

Since the number has period 3, I multiply both sides by 10^3:

$$x = (0.24\overline{122})_{10} = 0.24122122\ldots,$$

$$1000x = 241.22122122\ldots.$$

Next, subtract the first equation from the second:

$$999x = 240.98,$$

$$x = \frac{240.98}{999} = \frac{24098}{99900} = \frac{12049}{49950}. \quad \square$$

Continued Fractions

- A **finite continued fraction** is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{\ddots}{+\cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}.$$

  It can also be written as $[a_0; a_1, a_2, \ldots, a_n]$.

- Finite continued fractions with integer terms represent rational numbers.

- Every rational number can be expressed as a finite continued fraction using the Euclidean Algorithm, but the expression is not unique.

- The k^{th} **convergent** of $[a_0; a_1, a_2, \ldots, a_n]$ is (the value of) the continued fraction $[a_0; a_1, a_2, \ldots, a_k]$.

- The convergents *oscillate* about the value of the continued fraction.

- The convergents of a continued fraction may be computed from the continued fraction using a recursive algorithm. The algorithm produces rational numbers in lowest terms.

Definition. Let $a_0, \ldots a_n$ be real numbers, with $a_1, \ldots, a_n$ positive. An expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{\ddots}{+\cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

is a **finite continued fraction**.

To make the writing easier, I'll denote the continued fraction above by $[a_0; a_1, a_2, \ldots, a_n]$. In most cases, the a_i's will be integers.

Example.

$$\frac{47}{17} = 2 + \frac{13}{17} = 2 + \cfrac{1}{\cfrac{17}{13}} = 2 + \cfrac{1}{1 + \cfrac{4}{13}} = 2 + \cfrac{1}{1 + \cfrac{1}{\cfrac{13}{4}}} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4}}}.$$

In short form, $\dfrac{47}{17} = [2; 1, 3, 4]$.

A little bit of thought should convince you that you can express any rational number as a finite continued fraction in this way. In fact, the expansion corresponds to the steps in the Euclidean algorithm. First,

$$47 = 2 \cdot 17 + 13$$

$$17 = 1 \cdot 13 + 4$$

$$13 = 3 \cdot 4 + 1$$

$$4 = 4 \cdot 1 + 0$$

Rewrite these equations as

$$\frac{47}{17} = 2 + \frac{13}{17}$$

$$\frac{17}{13} = 1 + \frac{4}{13}$$

$$\frac{13}{4} = 3 + \frac{1}{4}$$

You can get the continued fraction I found above by substituting the third equation into the second, and then substituting the result into the first.

Since this *is* just the Euclidean algorithm, I can use the first two columns of the Extended Euclidean Algorithm table to get the numbers in the continued fraction expansion:

a	q
47	-
17	2
13	1
4	3
1	4

Notice that the successive quotients 2, 1, 3, and 4 are the numbers in the continued fraction expansion. □

Example. Find the finite continued fraction expansion for $\frac{117}{17}$.

a	q
117	-
17	6
15	1
2	7
1	2

$$\frac{117}{17} = [6; 1, 7, 2] = 6 + \cfrac{1}{1 + \cfrac{1}{7 + \cfrac{1}{2}}}. \quad \square$$

Lemma. Every finite continued fraction with integer terms represents a rational number.

Proof. If $a_0 \in \mathbb{Z}$, then $[a_0]$ is rational.

Inductively, suppose that a finite continued fraction with $n-1$ "levels" is a rational number. I want to show that

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{\ddots}{+\cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

is rational.

By induction,

$$x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{\ddots}{+\cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

is rational.

So

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{\ddots}{+\cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}} = a_0 + \frac{1}{x}$$

is the sum of two rational numbers, which is rational as well. $\square$

Example. The continued fraction expansion of a rational number is not unique. For example,

$$\frac{47}{17} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4}}} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{1}}}}.$$

And in general,

$$[a_0; a_1, a_2, \ldots, a_{n-1}, a_n] = [a_0; a_1, a_2, \ldots, a_{n-1}, a_n - 1, 1]. \quad \square$$

I want to talk about **infinite continued fractions** — things that look like

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{}{\ddots}}}}$$

In preparation for this, I'll look at the effect of *truncating* a continued fraction.

Definition. The k^{th} **convergent** of the continued fraction $[a_0; a_1, a_2, \ldots, a_n]$ is

$$c_k = [a_0; a_1, a_2, \ldots, a_k].$$

Note that for $k \geq n$, $c_k = [a_0; a_1, a_2, \ldots, a_n]$.

Example. $[1; 2, 3, 2]$

$$c_0 = 1$$

$$c_1 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$c_2 = 1 + \cfrac{1}{2 + \cfrac{1}{3}} = \frac{10}{7}$$

$$c_3 = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2}}} = \frac{23}{16}$$

And $c_4 = c_5 = \cdots = \dfrac{23}{16}$ as well. □

The next result gives an algorithm for computing the convergents of a continued fraction. It's important for theoretical reasons, too — I'll need it for several of the proofs that follow. For the theorem, I won't assume that the a_i's are integers, since I will need the general result later on.

Theorem. Let a_0, a_1, ..., a_n be positive real numbers. Let

$$p_0 = a_0, \quad q_0 = 1$$

$$p_1 = a_1 a_0 + 1, \quad q_1 = a_1$$

$$p_k = a_k p_{k-1} + p_{k-2}, \quad q_k = a_k q_{k-1} + q_{k-2}, \qquad k \geq 2.$$

Then the k^{th} convergent of $[a_0; a_1, a_2, \ldots, a_n]$ is $c_k = \dfrac{p_k}{q_k}$.

Proof. First, note that

$$[b_0; b_1, \ldots, b_k, b_{k+1}] = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{\ddots}{+\cfrac{1}{b_{k-1} + \cfrac{1}{b_k + \cfrac{1}{b_{k+1}}}}}}} = \left[b_0; b_1, \ldots, b_k + \frac{1}{b_{k+1}}\right]$$

by regarding the last two terms as a single term.

Note also that p_0, ..., p_{k-1} and q_0, ..., q_{k-1} are the same for these two fractions, since they only differ in the k^{th} term.

Now I'll start the proof — it will go by induction on k. For $k = 0$,

$$c_0 = a_0 = \frac{a_0}{1} = \frac{p_0}{q_0}.$$

And for $k = 1$,

$$c_1 = a_0 + \frac{1}{a_1} = \frac{a_1 a_0 + 1}{a_1} = \frac{p_1}{q_1}.$$

Suppose $k \ge 2$, and assume that result holds through the k^{th} convergent. Then

$$c_{k+1} = [a_0; a_1, \ldots, a_k, a_{k+1}] = \left[a_0; a_1, \ldots, a_k + \frac{1}{a_{k+1}}\right].$$

Now this is the k^{th} convergent of a continued fraction, so by induction this is $\dfrac{p_k}{q_k}$, where p_k and q_k refer to $\left[a_0; a_1, \ldots, a_k + \dfrac{1}{a_{k+1}}\right]$ (as opposed to $[a_0; a_1, \ldots, a_k, a_{k+1}]$). But what are the p_k and q_k for this fraction? They're given inductively by

$$p_k = (k\text{-th term})p_{k-1} + p_{k-2}, \quad q_k = (k\text{-th term})q_{k-1} + q_{k-2}.$$

Now p_{k-2}, p_{k-1}, q_{k-2}, q_{k-1} are the same for $\left[a_0; a_1, \ldots, a_k + \dfrac{1}{a_{k+1}}\right]$ and $[a_0; a_1, \ldots, a_k, a_{k+1}]$, as I noted at the start. On the other hand, the k^{th} term of $\left[a_0; a_1, \ldots, a_k + \dfrac{1}{a_{k+1}}\right]$ is $a_k + \dfrac{1}{a_{k+1}}$. So

$$c_{k+1} = \frac{\left(a_k + \dfrac{1}{a_{k+1}}\right)p_{k-1} + p_{k-2}}{\left(a_k + \dfrac{1}{a_{k+1}}\right)q_{k-1} + q_{k-2}} = \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}} = \frac{a_{k+1}p_k + p_{k-1}}{a_{k+1}q_k + q_{k-1}} = \frac{p_{k+1}}{q_{k+1}}.$$

(The next to the last equality also follows by induction.) This shows that the result holds for $k + 1$, so the induction step is complete. ☐

Example. $[1; 2, 1, 2, 1]$

a_k	p_k	q_k	c_k
1	1	1	1
2	3	2	$\frac{3}{2} = 1.5$
1	4	3	$\frac{4}{3} \approx 1.33333$
2	11	8	$\frac{11}{8} = 1.375$
1	15	11	$\frac{15}{11} \approx 1.36364$

There is a pattern to the computation of the p's and q's which makes things pretty easy. To get the next p, for instance, multiply the current a by the last p and add the next-to-the-last p.

a	p	q
1	1	1
2	3	2
1		
2		
1		

Fill in the a's, p_0, q_0, p_1, and q_1.

a	p	q
1	1	1
2	3	2
1	4	3
2		
1		

$p_2 = (1)(3) + 1 = 4$
$q_2 = (1)(2) + 1 = 3$

a	p	q
1	1	1
2	3	2
1	4	3
2	11	8
1		

$p_3 = (2)(4) + 3 = 11$
$q_3 = (2)(3) + 2 = 8$

a	p	q
1	1	1
2	3	2
1	4	3
2	11	8
1	15	11

$p_4 = (1)(11) + 4 = 15$
$q_4 = (1)(8) + 3 = 11$

Notice that the convergents oscillate, and that the fractions which give the convergents are always in lowest terms. □

Example. $[1; 1, 3, 1, 3]$

a_k	p_k	q_k	c_k
1	1	1	1
1	2	1	2
3	7	4	$\frac{7}{4} = 1.75$
1	9	5	$\frac{9}{5} = 1.8$
3	34	19	$\frac{34}{19} \approx 1.78947$

Again, notice that the convergents oscillate, and that the fractions for the convergents are always in lowest terms. □

I'll prove that the convergent fractions are in lowest terms first.

Theorem. Let a_0, a_1, ..., a_n be positive real numbers. Let

$$p_0 = a_0, \quad q_0 = 1$$

$$p_1 = a_1 a_0 + 1, \quad q_1 = a_1$$

$$p_k = a_k p_{k-1} + p_{k-2}, \quad q_k = a_k q_{k-1} + q_{k-2}, \qquad k \geq 2.$$

Then

$$p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}.$$

Corollary. Let $a_0, a_1, \ldots, a_n$ be positive integers. Let

$$p_0 = a_0, \quad q_0 = 1$$
$$p_1 = a_1 a_0 + 1, \quad q_1 = a_1$$
$$p_k = a_k p_{k-1} + p_{k-2}, \quad q_k = a_k q_{k-1} + q_{k-2}, \qquad k \ge 2.$$

For $k \ge 1$, $\dfrac{p_k}{q_k}$ is in lowest terms.

Proof. $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1} = \pm 1$ implies that $(p_k, q_k) = 1$. □

Proof of the Theorem. I'll induct on k. For $k = 1$,

$$p_1 q_0 - p_0 q_1 = (a_1 a_0 + 1)(1) - (a_0)(a_1) = 1 = 1^{1-1}.$$

Take $k > 1$, and assume the result holds for k. Then

$$p_{k+1} q_k - p_k q_{k+1} = (a_{k+1} p_k + p_{k-1}) q_k - p_k (a_{k+1} q_k + q_{k-1}) = p_{k-1} q_k - p_k q_{k-1} =$$
$$-\left(p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}\right) = -(-1)^{k-1} = (-1)^k.$$

This proves the result for $k + 1$, so the general result is true by induction. □

Example. I'll show later that $\dfrac{1+\sqrt{5}}{2}$ (the golden ratio) has the *infinite* continued fraction expansion $[1; 1, 1, \ldots]$. Here are the first ten convergents:

a_k	p_k	q_k	c_k
1	1	1	1
1	2	1	2
1	3	2	1.5
1	5	3	1.66667
1	8	5	1.6
1	13	8	1.625
1	21	13	1.61538
1	34	21	1.61905
1	55	34	1.61765
1	89	55	1.61818

In fact, $\dfrac{1+\sqrt{5}}{2} \approx 1.61803$. In this case, you can see formally that $[1; 1, 1, \ldots]$ should be $\dfrac{1+\sqrt{5}}{2}$. Let

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

Notice that x contains a copy of itself as the bottom of the first fraction! So

$$x = 1 + \frac{1}{x}, \quad x^2 = x + 1, \quad x^2 - x - 1 = 0.$$

The roots are $\dfrac{1 \pm \sqrt{5}}{2}$. Since the fraction is positive, take the positive root to obtain $x = \dfrac{1+\sqrt{5}}{2}$. □
