Infinite Continued Fractions

- The value of an infinite continued fraction $[a_0; a_1, a_2, \ldots]$ is
$$\lim_{k\to\infty} c_k, \quad \text{where} \quad c_k \quad \text{is the } k\text{-th convergent.}$$
- If $[a_0; a_1, a_2, \ldots]$ is an infinite continued fraction with positive terms, then $\lim_{k\to\infty} c_k$ exists.
- The convergents of an infinite continued fraction with positive terms converge by oscillation to its value.
- If $[a_0; a_1, a_2, \ldots]$ is an infinite continued fraction with positive terms, then its value is an irrational number.
- Every irrational number x has a unique infinite continued fraction expansion $[a_0; a_1, a_2, \ldots]$ whose terms are given recursively by
$$x_0 = x, \quad \text{and} \quad a_k = [x_k], \quad x_{k+1} = \frac{1}{x_k - a_k} \text{ for } k \geq 1.$$

If $[a_0; a_1, a_2, \ldots]$ is an infinite continued fraction, I want to define its *value* to be
$$\lim_{k\to\infty} c_k, \quad \text{where} \quad c_k \quad \text{is the } k\text{-th convergent.}$$

For this to make sense, I need to show that this limit exists.
In what follows, take as given an infinite continued fraction $[a_0; a_1, a_2, \ldots]$. Define
$$p_0 = a_0, \quad q_0 = 1$$
$$p_1 = a_1 a_0 + 1, \quad q_1 = a_1$$
$$p_k = a_k p_{k-1} + p_{k-2}, \quad q_k = a_k q_{k-1} + q_{k-2}, \qquad k \geq 2,$$
$$c_k = \frac{p_k}{q_k}.$$

Lemma.

(a) $c_k - c_{k-1} = \dfrac{(-1)^{k-1}}{q_{k-1} q_k}$

(b) $c_k - c_{k-2} = \dfrac{a_k (-1)^k}{q_{k-2} q_k}$

Proof. For the first part,
$$c_k - c_{k-1} = \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{p_k q_{k-1} - p_{k-1} q_k}{q_{k-1} q_k} = \frac{(-1)^{k-1}}{q_{k-1} q_k}.$$

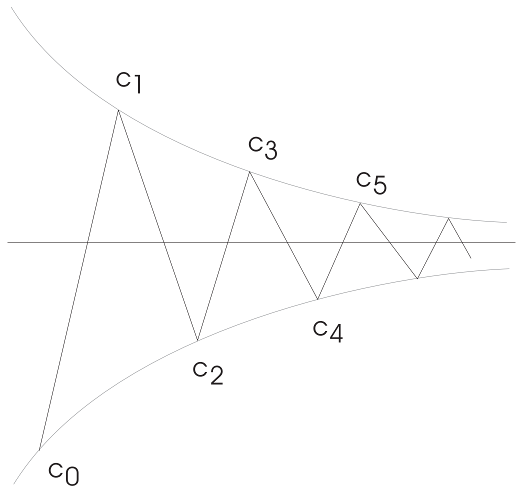
For the second part,
$$c_k - c_{k-2} = \frac{p_k}{q_k} - \frac{p_{k-2}}{q_{k-2}} = \frac{p_k q_{k-2} - p_{k-2} q_k}{q_{k-2} q_k} = \frac{(a_k p_{k-1} + p_{k-2}) q_{k-2} - p_{k-2}(a_k q_{k-1} + q_{k-2})}{q_{k-2} q_k} =$$

$$\frac{a_k(p_{k-1}q_{k-2} - p_{k-2}q_{k-1})}{q_{k-2}q_k} = \frac{a_k \cdot (-1)^{k-2}}{q_{k-2}q_k} = \frac{a_k(-1)^k}{q_{k-2}q_k}. \quad \square$$

Lemma.

$$c_1 > c_3 > c_5 > \cdots > c_4 > c_2 > c_0.$$

That is, the odd convergents get smaller, the even convergents get bigger, and any odd convergent is bigger than any even convergent.



Proof. If k is even, then

$$c_k - c_{k-2} = \frac{a_k(-1)^k}{q_{k-2}q_k} > 0, \quad \text{so} \quad c_k > c_{k-2}.$$

This shows that the even terms get bigger.

If k is odd, then

$$c_k - c_{k-2} = \frac{a_k(-1)^k}{q_{k-2}q_k} < 0, \quad \text{so} \quad c_k < c_{k-2}.$$

This shows that the odd terms get smaller.

Finally, I have to show any odd term is bigger than any even term. Note that it's true if the terms are adjacent:

$$c_{2n+1} - c_{2n} = \frac{(-1)^{(2n+1)-1}}{q_{k-1}q_k} > 0, \quad \text{so} \quad c_{2n+1} > c_{2n}.$$

Next, do the general case. Let c_{2n+1} be an odd term and let c_{2m} be an even term. Then

$$c_{2n+1} > c_{2n+2m+1} > c_{2n+2m} > c_{2m}.$$

The first inequality is true because odd terms decrease. The second inequality is true because I just observed that an odd term is bigger than an *adjacent* even. Finally, the last inequality is true because even terms increase.

I've shown that any odd term is bigger than any even term, and that completes the proof. $\square$

Example. The first ten convergents for the golden ratio $\dfrac{1+\sqrt{5}}{2}$ are:

$$\underset{c_1}{2} > \underset{c_3}{1.66667} > \underset{c_5}{1.625} > \underset{c_7}{1.61905} > \underset{c_9}{1.61818}$$

$$\underset{c_8}{1.61765} > \underset{c_6}{1.61538} > \underset{c_4}{1.6} > \underset{c_2}{1.5} > \underset{c_0}{1}$$

The odd convergents get smaller, the even convergents get bigger, and any odd convergent is bigger than any even convergent. □

Lemma. $q_k \geq k$ for all $k \geq 1$.

Proof. I'll induct on k. $q_1 = a_1 = 1$, so the result holds for $k = 1$. Take $k > 1$, and assume it holds for numbers $\leq k$. I'll prove that it holds for $k + 1$.

$$q_{k+1} = a_{k+1}q_k + q_{k-1} \geq a_{k+1} \cdot k + (k-1) \geq 1 \cdot k + (k-1) = 2k - 1.$$

But $k > 1$ means $k \geq 2$, so $2k - 1 \geq k + 1$, and hence $q_{k+1} \geq k + 1$. This completes the induction step. □

Theorem. Let $[a_0; a_1, a_2, \ldots]$ be an infinite continued fraction with $a_k > 0$ for $k \geq 1$, and let c_k be the k-th convergent. Then

$$\lim_{k\to\infty} c_k \quad \text{exists.}$$

Proof.

$$c_1 > c_3 > c_5 > \cdots$$

is a decreasing sequence of numbers, and it's bounded below — by any even convergent, for example. A standard result from analysis (see, for example, Theorem 3.14 of [1]) asserts that such a sequence must have a limit, so

$$\lim_{k\to\infty} c_{2k+1} \quad \text{exists.}$$

Likewise,

$$\cdots > c_4 > c_2 > c_0$$

is an increasing sequence of numbers that's bounded above — by any odd convergent, for example. The result from analysis mentioned above says that the sequence has a limit:

$$\lim_{k\to\infty} c_{2k} \quad \text{exists.}$$

I have to show that the two limits agree.

$$0 \leq c_{2k+1} - c_{2k} = \frac{(-1)^{2k+1-1}}{q_{2k}q_{2k+1}} \leq \frac{1}{(2k)(2k+1)},$$

since the previous lemma implies that $q_{2k} \geq 2k$ and $q_{2k+1} \geq 2k + 1$.

Now let $k \to \infty$. $\dfrac{1}{(2k)(2k+1)} \to 0$, so by the Squeezing Theorem of calculus,

$$\lim_{k\to\infty} (c_{2k+1} - c_{2k}) = 0, \text{ i.e. } \lim_{k\to\infty} c_{2k+1} = \lim_{k\to\infty} c_{2k}.$$

Since the odd and even terms approach the same limit, $\lim_{k\to\infty} c_k$ exists. □

Knowing this, I'm justified in *defining*

$$[a_0; a_1, a_2, \ldots] = \lim_{k\to\infty} c_k.$$

What can I say about its value?

Theorem. Let $[a_0; a_1, a_2, \ldots]$ be an infinite continued fraction with $a_k > 0$ for $k \geq 1$. Then $[a_0; a_1, a_2, \ldots]$ is irrational.

Proof. Write $x = [a_0; a_1, a_2, \ldots]$ for short. I want to show that x is irrational. Suppose on the contrary that $x = \dfrac{p}{q}$, where p and q are integers. I will show this leads to a contradiction.

Since the odd convergents are bigger than x and the even convergents are smaller than x,

$$c_{2k+1} > x > c_{2k}.$$

Then

$$\begin{aligned}
c_{2k+1} - c_{2k} &> x - c_{2k} > 0,\\
\frac{(-1)^{2k}}{q_{2k}q_{2k+1}} &> x - c_{2k} > 0,\\
\frac{1}{q_{2k}q_{2k+1}} &> x - c_{2k} > 0,\\
\frac{1}{q_{2k}q_{2k+1}} &> x - \frac{p_{2k}}{q_{2k}} > 0,\\
\frac{1}{q_{2k+1}} &> xq_{2k} - p_{2k} > 0,\\
\frac{1}{q_{2k+1}} &> \frac{pq_{2k}}{q} - p_{2k} > 0,\\
\frac{q}{q_{2k+1}} &> pq_{2k} - p_{2k}q > 0.
\end{aligned}$$

Notice that this inequality is true for all k, and that the junk in the middle is an *integer*. But q is fixed, and $q_{2k+1} \geq 2k+1$, so if I make k sufficiently large eventually q_{2k+1} will become bigger than q. Then $\dfrac{q}{q_{2k+1}}$ will be a fraction less than 1, and I have an *integer* $pq_{2k} - p_{2k}q$ caught between 0 and a fraction less than 1. Since this is impossible, x can't be rational. □

Now I know that every infinite continued fraction made of positive integers represents an irrational number. The converse is also true, and the next result gives an algorithm for computing the continued fraction expansion.

Theorem. Let $x \in \mathbb{R}$ be irrational. Let $x_0 = x$, and

$$a_k = [x_k], \quad x_{k+1} = \frac{1}{x_k - a_k} \text{ for } k \geq 0.$$

Then

$$x = [a_0; a_1, a_2, \ldots].$$

Proof.

Step 1. x_k is irrational for $k \geq 0$.

Since x is irrational and $x_0 = x$, the result is true for $k = 0$.

Assume that $k > 0$ and that the result is true for $k-1$. I want to show that x_k is irrational.

Suppose on the contrary that $x_k = \dfrac{s}{t}$, where $s, t \in \mathbb{Z}$. Then

$$\frac{s}{t} = \frac{1}{x_{k-1} - a_{k-1}} \quad \text{so} \quad x_{k-1} = a_{k-1} + \frac{t}{s}.$$

Now all the a_k's are clearly integers (since $a_k = [x_k]$ means they're outputs of the greatest integer function), so $a_{k-1} + \dfrac{t}{s}$ is the sum of an integer and a rational number. Therefore, it's rational, so x_{k-1} is rational, contrary to the induction hypothesis.

It follows that x_k is irrational. By induction, x_k is irrational for all $k \geq 0$.

Step 2. The a_k's are positive integers for $k \geq 1$.

I already observed that the a_k's are integers.
Let $k \geq 0$. Since $a_k = [x_k]$, the definition of the greatest integer function gives

$$a_k \leq x_k < a_k + 1.$$

But x_k is irrational, so $a_k \neq x_k$. Hence,

$$\begin{gathered} a_k < x_k < a_k + 1, \\ 0 < x_k - a_k < 1, \\ x_{k+1} = \frac{1}{x_k - a_k} > 1, \\ a_{k+1} = [x_{k+1}] \geq 1. \end{gathered}$$

Since $k \geq 0$, this proves that the a_k's are positive integers for $k \geq 1$.

Step 3.

$$\lim_{k \to \infty} c_k = \lim_{k \to \infty} [a_0; a_1, \ldots, a_k] = x.$$

First, I'll get a formula for x in terms of the p's, q's, and a's.
Then I'll find $\left|x - \dfrac{p_k}{q_k}\right|$ and show that it's less than something which goes to 0.
To get the formula for x, start with

$$x_{k+1} = \frac{1}{x_k - a_k}.$$

Do some algebra to get

$$x_k = a_k + \frac{1}{x_{k+1}}.$$

Write out this equation for a few values of k:

$$\begin{gathered} x_0 = a_0 + \frac{1}{x_1} \\ x_1 = a_1 + \frac{1}{x_2} \\ x_2 = a_2 + \frac{1}{x_3} \\ x_3 = a_3 + \frac{1}{x_4} \\ \vdots \end{gathered}$$

Substituting the second equation of the set into the first gives

$$x_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{x_2}}.$$

Substituting $x_2 = a_2 + \dfrac{1}{x_3}$ into this equation gives

$$x_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{x_3}}}.$$

Substituting $x_3 = a_3 + \dfrac{1}{x_4}$ into this equation gives

$$x_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{x_4}}}}.$$

You get the idea. In general,

$$x = x_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_k + \cfrac{1}{x_{k+1}}}}} = [a_0; a_1, a_2, \ldots, a_k, x_{k+1}].$$

Recall the recursion formulas for convergents:

$$p_k = a_k p_{k-1} + p_{k-2} \quad \text{and} \quad q_k = a_k q_{k-1} + q_{k-2}.$$

The right sides only involve terms up to a_k (and p's and q's of smaller indices). Therefore, the fractions

$$[a_0; a_1, a_2, \ldots, a_k, x_{k+1}] \quad \text{and} \quad [a_0; a_1, a_2, \ldots, a_k, a_{k+1}, \ldots]$$

have the same p's and q's through index k.

Using the recursion formula for convergents, I get

$$x = x_0 = [a_0; a_1, a_2, \ldots, a_k, x_{k+1}] = \frac{x_{k+1}p_k + p_{k-1}}{x_{k+1}q_k + q_{k-1}}.$$

Therefore,

$$x - \frac{p_k}{q_k} = \frac{x_{k+1}p_k + p_{k-1}}{x_{k+1}q_k + q_{k-1}} - \frac{p_k}{q_k} = \frac{x_{k+1}p_kq_k + p_{k-1}q_k - x_{k+1}p_kq_k - p_kq_{k-1}}{(x_{k+1}q_k + q_{k-1})q_k} =$$

$$\frac{p_{k-1}q_k - p_kq_{k-1}}{(x_{k+1}q_k + q_{k-1})q_k} = \frac{(-1)^k}{(x_{k+1}q_k + q_{k-1})q_k}.$$

Take absolute values:

$$\left|x - \frac{p_k}{q_k}\right| = \frac{1}{(x_{k+1}q_k + q_{k-1})q_k}.$$

Now

$$x_{k+1} > [x_{k+1}] = a_{k+1}, \quad \text{so} \quad x_{k+1}q_k + q_{k-1} > a_{k+1}q_k + q_{k-1} = q_{k+1}.$$

Therefore,

$$\frac{1}{x_{k+1}q_k + q_{k-1}} < \frac{1}{q_{k+1}},$$

$$\frac{1}{(x_{k+1}q_k + q_{k-1})q_k} < \frac{1}{q_{k+1}q_k},$$

$$\left|x - \frac{p_k}{q_k}\right| < \frac{1}{q_{k+1}q_k}.$$

By an earlier lemma, $q_k \geq k$ and $q_{k+1} \geq k+1$, so

$$\left|x - \frac{p_k}{q_k}\right| < \frac{1}{q_{k+1}q_k} \leq \frac{1}{k(k+1)}.$$

Now $\displaystyle\lim_{k\to\infty} \frac{1}{k(k+1)} = 0$, so by the Squeezing Theorem

$$\lim_{k\to\infty} \left|x - \frac{p_k}{q_k}\right| = 0.$$

This implies that

$$\lim_{k\to\infty} \frac{p_k}{q_k} = x. \quad \square$$

Example. I'll compute the continued fraction expansion of π. Here are the first two steps:

$$x_0 = \pi, \quad a_0 = [x_0] = [\pi] = 3$$

$$x_1 = \frac{1}{x_0 - a_0} \approx 7.06251, \quad a_1 = [x_1] = 7$$

Continuing in this way, I obtain:

a_k	p_k	q_k	c_k
3	3	1	3
7	22	7	$\frac{22}{7}$
15	333	106	$\frac{333}{106}$
1	355	113	$\frac{355}{113}$
292	103993	33102	$\frac{103993}{33102}$

$\square$

Example. I'll compute the continued fraction expansion of $\sqrt{5}$:

$$x_0 = \sqrt{5} \quad a_0 = [\sqrt{5}] \approx [2.23607] = 2$$

$$x_1 = \frac{1}{\sqrt{5} - 2}, \quad a_1 = \left[\frac{1}{\sqrt{5} - 2}\right] \approx [4.23607] = 4$$

$$x_2 = \frac{1}{\dfrac{1}{\sqrt{5} - 2} - 4} = \frac{2 - \sqrt{5}}{4\sqrt{5} - 9}, \quad a_2 = \left[\frac{2 - \sqrt{5}}{4\sqrt{5} - 9}\right] \approx [4.23607] = 4$$

In fact, the continued fraction expansion for $\sqrt{5}$ is $[2; 4, 4, 4, \ldots]$. $\square$

Theorem. The continued fraction expansion of an irrational number is unique.

Proof. Suppose

$$[a_0; a_1, a_2, \ldots] = x = [b_0; b_1, b_2, \ldots]$$

are two continued fractions for the irrational number x, where $a_k, b_k \in \mathbb{Z}$ and $a_k, b_k \geq 1$ for $k \geq 1$. I want to show that $a_k = b_k$ for all k.

Recall that:

- The even convergents are smaller than x.
- The odd convergents are greater than x.

Therefore,

$$a_0 < x < a_0 + \frac{1}{a_1}.$$

Now

$$\begin{aligned} a_1 &\geq 1 \\ \frac{1}{a_1} &\leq 1 \\ a_0 + \frac{1}{a_1} &\leq a_0 + 1 \\ x &< a_0 + 1 \end{aligned}$$

Thus, a_0 *is an integer less than* x*, and the next larger integer* $a_0 + 1$ *is greater than* x. This means that $a_0 = [x]$.

The same reasoning applies to the b's. Therefore, $b_0 = [x]$, so $a_0 = b_0$.

Hence,

$$\begin{gathered} [a_0; a_1, a_2, \ldots] = [b_0; b_1, b_2, \ldots] \\ a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ddots}} \\ \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ddots}} \\ [a_1; a_2, a_3, \ldots] = [b_1; b_2, b_3, \ldots] \end{gathered}$$

I can continue in the same way to show that $a_k = b_k$ for all k. □

Here's a summary of some of the important results on infinite continued fractions:

1. An irrational number has a unique infinite continued fraction expansion.

2. The algorithm for computing the continued fraction expansion of an irrational number x is:

$$x_0 = x, \quad \text{and} \quad a_k = [x_k], \quad x_{k+1} = \frac{1}{x_k - a_k} \quad \text{for} \quad k \geq 1.$$

Then

$$x = [a_0; a_1, a_2, \ldots].$$

3. If $[a_0; a_1, a_2, \ldots]$ is the continued fraction expansion of an irrational number, then a_k is a positive integer for $k \geq 1$.

4. If $x = [a_0; a_1, a_2, \ldots]$ is the continued fraction expansion of an irrational number and $\{p_k\}$ and $\{q_k\}$ are defined by the recursion formulas for convergents, then

$$\left| x - \frac{p_k}{q_k} \right| < \frac{1}{q_{k+1} q_k}.$$

Example. Here is the continued fraction expansion for $e + \pi$.

x_k	a_k	p_k	q_k	c_k
5.85987	5	5	1	5
1.16296	1	6	1	6
6.13646	6	41	7	$\frac{41}{7}$
7.32821	7	293	50	$\frac{293}{50}$
3.0468	3	920	157	$\frac{920}{157}$
21.3697	21	19613	3347	$\frac{19613}{3347}$

Now

$$e + \pi - \frac{920}{157} \approx 0.000001871 \quad \text{while} \quad \frac{1}{157 \cdot 3347} \approx 0.000001903.$$

Thus, in this case,

$$\left| e + \pi - \frac{920}{157} \right| < \frac{1}{157 \cdot 3347}. \quad \square$$

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[1] Walter Rudin, *Principles of Mathematical Analysis* (3$^{\text{rd}}$ edition). New York: McGraw-Hill Book Company, 1976.

Periodic Continued Fractions

- A **quadratic irrational** is an irrational number which is a root of a quadratic equation with integer coefficients.
- Quadratic irrationals can be expressed in the form $\dfrac{p+\sqrt{q}}{r}$, where $p, q, r \in \mathbb{Z}$, $r \neq 0$, and q is positive and not a perfect square.
- Quadratic irrationals are exactly the real numbers which have infinite *periodic* continued fraction expansions.

Definition. A **quadratic irrational** is an irrational number which is a root of a quadratic equation

$$ax^2 + bx + c = 0, \quad a, b, c \in \mathbb{Z}, \quad a \neq 0.$$

Lemma. A number is a quadratic irrational if and only if it can be written in the form $\dfrac{p+\sqrt{q}}{r}$, where $p, q, r \in \mathbb{Z}$, $r \neq 0$, and q is positive and not a perfect square.

Proof. Suppose x is a quadratic irrational. Then x is a root of

$$ax^2 + bx + c = 0, \quad a, b, c \in \mathbb{Z}, a \neq 0.$$

By the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$-b$, $b^2 - 4ac$, and $2a$ are integers, and $2a \neq 0$, since $a \neq 0$.
If $b^2 - 4ac = 0$, then $x = -\dfrac{b}{2a}$, which is a rational number, contrary to assumption.
If $b^2 - 4ac < 0$, then x is complex, again contrary to assumption.
Hence, $b^2 - 4ac > 0$.

Finally, if $b^2 - 4ac$ is a perfect square, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is rational. Hence, $b^2 - 4ac$ is not a perfect square.

For the converse, suppose $x = \dfrac{p+\sqrt{q}}{r}$, where $p, q, r \in \mathbb{Z}$, $r \neq 0$, and q is positive and not a perfect square. Then

$$rx - p = \sqrt{q}, \quad (rx - p)^2 = q, \quad r^2x^2 - 2rpx + (p^2 - q) = 0.$$

This is a quadratic equation with integer coefficients, and $r^2 \neq 0$ since $r \neq 0$. Therefore, x is a quadratic irrational. □

Theorem. (Lagrange) The quadratic irrationals are exactly the real numbers which can be represented by infinite periodic continued fractions.

I'm going to prove one direction — that periodic continued fractions are quadratic irrationals. I need a series of lemmas; the lemmas are motivated by the informal procedure of the following example.

Example. Consider $x = [5; 2, 1, 2, 2, 1, 2, 2, \ldots] = [5; 2, \overline{1, 2, 2}]$.
I'll write x in closed form. Let $y = [\overline{1, 2, 2}]$. Then

$$x = 5 + \cfrac{1}{2 + \cfrac{1}{y}}.$$

On the other hand,

$$y = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{y}}}.$$

After some simplification, I get

$$5y^2 - 5y - 3 = 0, \quad y = \frac{5 \pm \sqrt{37}}{10}.$$

y must be positive, so $y = \dfrac{5 + \sqrt{37}}{10}$. Therefore,

$$x = 5 + \cfrac{1}{2 + \cfrac{1}{\cfrac{5 + \sqrt{37}}{10}}} = \frac{643 + 5\sqrt{37}}{126}. \quad \square$$

The idea of the lemmas is simply to emulate the algebra I just did.

Lemma 1. If x is a quadratic irrational and a_0 is an integer, then $a_0 + \dfrac{1}{x}$ is a quadratic irrational.

Proof. Write $x = \dfrac{a + \sqrt{b}}{c}$, where $a, b, c \in \mathbb{Z}$, $c \neq 0$, and b is positive and not a perfect square. Then

$$a_0 + \frac{1}{x} = a_0 + \cfrac{1}{\cfrac{a + \sqrt{b}}{c}} = a_0 + \frac{c}{a + \sqrt{b}} = \frac{(a_0 a^2 + ac - a_0 b) - c\sqrt{b}}{a^2 - b}.$$

(I've suppressed the ugly algebra involved in combining the fractions and rationalizing the denominator.) The last expression is a quadratic irrational; note that $a^2 - b \neq 0$, because b is not a perfect square. $\square$

Lemma 2. If x is a quadratic irrational and $a_0, a_1, \ldots, a_n$ are integers, then

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{}{\ddots + \cfrac{1}{a_n + \cfrac{1}{x}}}}}$$

is a quadratic irrational.

Proof. I'll use induction. The case $n = 0$ was done in Lemma 1.

Suppose $n > 0$, and suppose the result is true for $n - 1$. Then in

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{}{\ddots + \cfrac{1}{a_n + \cfrac{1}{x}}}}}, \quad \text{the subfraction} \quad a_1 + \cfrac{1}{a_2 + \cfrac{}{\ddots + \cfrac{1}{a_n + \cfrac{1}{x}}}}$$

is a quadratic irrational by the induction hypothesis.

But the original fraction is just $a_0 + \dfrac{1}{\text{(the subfraction)}}$, so it's a quadratic irrational by Lemma 1. This completes the induction step, so the result is true for all $n \geq 0$. $\square$

Lemma 3. Let $a_0, a_1, \ldots, a_n \in \mathbb{Z}$. Then

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{}{\ddots + \cfrac{1}{a_n + \cfrac{1}{x}}}}}, \quad \text{can be written as} \quad \frac{ax+b}{cx+d},$$

where $a, b, c, d \in \mathbb{Z}$.

Proof. Your experience with algebra should tell you this is obvious, but I'll give the proof by induction anyway.

For $n = 0$, I have

$$a_0 + \frac{1}{x} = \frac{a_0 x + 1}{x}.$$

This has the right form.

Take $n > 0$, and assume the result is true for $n - 1$. Then in

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{}{\ddots + \cfrac{1}{a_n + \cfrac{1}{x}}}}}, \quad \text{the subfraction} \quad a_1 + \cfrac{1}{a_2 + \cfrac{}{\ddots + \cfrac{1}{a_n + \cfrac{1}{x}}}}$$

can be written as $\dfrac{ax+b}{cx+d}$, $a, b, c, d \in \mathbb{Z}$, by induction.

The original fraction is therefore

$$a_0 + \cfrac{1}{\cfrac{ax+b}{cx+d}} = \frac{(a_0 a + c)x + (a_0 b + d)}{ax+b}.$$

(I've suppressed some easy but ugly algebra again.) The last fraction is in the right form, so this completes the induction step. The result is therefore true for all $n \geq 0$. $\square$

I'm ready to prove that periodic continued fractions are quadratic irrationals. First, I'll consider those that start repeating immediately.

Lemma 4. If $a_0, a_1, \ldots, a_n \in \mathbb{Z}$, then

$$x = [\overline{a_0; a_1, \ldots, a_n}]$$

is a quadratic irrational.

Proof. First, x is irrational, because it is an *infinite* continued fraction.

By Lemma 3,

$$x = [\overline{a_0; a_1, \ldots, a_n}] = [a_0; a_1, \ldots, a_n, x] = \frac{ax+b}{cx+d},$$

where $a, b, c, d \in \mathbb{Z}$.

Hence,

$$cx^2 + dx = ax + b, \quad cx^2 + (d - a)x - b = 0.$$

Therefore, x is a quadratic irrational. □

In the general case, the fraction does not start repeating immediately.

Proposition. If $b_0, b_1, \ldots, b_m, a_0, a_1, \ldots, a_n \in \mathbb{Z}$, then

$$x = [b_0; b_1, \ldots, b_m, \overline{a_0, a_1, \ldots, a_n}]$$

is a quadratic irrational.

Proof. $[\overline{a_0, a_1, \ldots, a_n}]$ is a quadratic irrational by Lemma 4. Therefore,

$$x = [b_0; b_1, \ldots, b_m, x]$$

is a quadratic irrational by Lemma 2. □

The converse states the quadratic irrationals give rise to periodic continued fractions. I won't give the proof; however, here's an example which shows how you can go from a quadratic equation to a periodic continued fraction (at least in this case).

Example. Suppose x is a quadratic irrational satisfying $x^2 + x - 1 = 0$. Rewrite the equation as

$$x(x + 1) - 1 = 0, \quad \text{and then} \quad x = \frac{1}{1 + x}.$$

Now substitute $x = \dfrac{1}{1 + x}$ for x *in the right side*:

$$x = \cfrac{1}{1 + \cfrac{1}{1 + x}}.$$

Do it again:

$$x = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + x}}}.$$

It's clear that you can keep going, and so $x = [0; \overline{1}]$. □

Rational Approximation by Continued Fractions

- The convergents of a continued fraction expansion of x give the **best rational approximations** to x. Specifically, the only way a fraction can approximate x *better than* a convergent is if the fraction has a bigger denominator than the convergent.

The first lemma says that the denominators of convergents of continued fractions increase.

Lemma. Let a_0, a_1, a_2, ... be a sequence of integers, where $a_k > 0$ for $k \geq 1$. Define

$$p_0 = a_0, \quad q_0 = 1$$

$$p_1 = a_1 a_0 + 1, \quad q_1 = a_1$$

$$p_k = a_k p_{k-1} + p_{k-2}, \quad q_k = a_k q_{k-1} + q_{k-2}, \qquad k \geq 2.$$

Then $q_{k+1} > q_k$ for $k > 0$.

Proof. Let $k > 0$. Note that q_{k-1} is a positive integer. So

$$q_{k+1} = a_{k+1} q_k + q_{k-1} > a_{k+1} q_k \geq 1 \cdot q_k = q_k,$$

where $a_{k+1} \geq 1$ because the a's are positive integers from a_1 on. □

The convergents of a continued fraction oscillate around the limiting value, and the convergents are always fractions in lowest terms. In fact, the convergents are the *best rational approximations* to the value of the continued fraction. I'll state the precise result without proof.

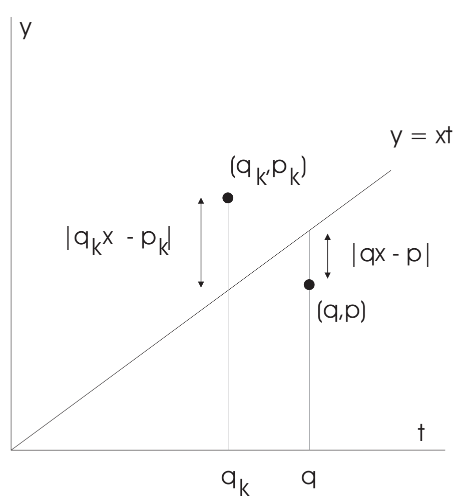
Theorem. Let x be irrational, and let $c_k = \dfrac{p_k}{q_k}$ be the k-th convergent in the continued fraction expansion of x. Suppose $p, q \in \mathbb{Z}$, $q > 0$, and

$$|qx - p| < |q_k x - p_k|.$$

Then $q \geq q_{k+1}$. □

Here's what the result means. Draw the line through the origin in the t-y plane with slope x. Plot the points (p, q) and (p_k, q_k).

The hypothesis $|qx - p| < |q_k x - p_k|$ says that the vertical distance from (q, p) to $y = xt$ is less than the vertical distance from (q_k, p_k) to $y = xt$.



The conclusion says that $q \geq q_{k+1}$. In fact, since $q_{k+1} > q_k$, $q > q_k$: The denominator of $\dfrac{p}{q}$ is bigger than that of $\dfrac{p_k}{q_k}$.

In other words, the only way the point (p, q) can be closer to the line is if its y-coordinate is bigger.

I can restate the theorem in the form of a corollary in which you can see the fractions in question approximating x.

Corollary. Let x be irrational, and let $c_k = \dfrac{p_k}{q_k}$ be the k-th convergent in the continued fraction expansion of x. Suppose $p, q \in \mathbb{Z}$, $q > 0$, and

$$\left|x - \frac{p}{q}\right| < \left|x - \frac{p_k}{q_k}\right|.$$

Then $q > q_k$.

Proof. Given the hypotheses of the corollary, suppose on the contrary that $q \leq q_k$. Since

$$\left|x - \frac{p}{q}\right| < \left|x - \frac{p_k}{q_k}\right|,$$

I can multiply the two inequalities to get

$$|qx - p| < |q_k x - p_k|.$$

Apply the theorem to obtain $q \geq q_{k+1}$. But then $q_k \geq q \geq q_{k+1}$, which contradicts the fact that the q's increase.

Therefore, $q > q_k$. □

This result says that the only way a rational number $\dfrac{p}{q}$ can approximate a continued fraction *better than* a convergent $\dfrac{p_k}{q_k}$ is if the fraction has a bigger denominator than the convergent.

Example. Here are the convergents for the continued fraction expansion for π:

a_k	p_k	q_k	c_k
3	3	1	3
7	22	7	$\dfrac{22}{7}$
15	333	106	$\dfrac{333}{106}$
1	355	113	$\dfrac{355}{113}$
292	103993	33102	$\dfrac{103993}{33102}$

$\dfrac{355}{113} \approx 3.141592920$, which is in error in the seventh place. The theorem says that a fraction $\dfrac{p}{q}$ can be closer to π than $\dfrac{355}{113}$ only if $q > 113$. □

The next result is sort of a converse to the previous two results. It says that if a rational number approximates an irrational number x "sufficiently well", then the rational number must be a convergent in the continued fraction expansion for x.

Theorem. Let x be irrational, and let $\frac{p}{q}$ be a rational number in lowest terms with $q > 0$. Suppose that

$$\left|x - \frac{p}{q}\right| < \frac{1}{2q^2}.$$

Then $\frac{p}{q}$ is a convergent in the continued fraction expansion for x.

Proof. Since $q_k \ge k$ for $k \ge 0$, the q's form a strictly increasing sequence of positive integers. Therefore, for some k,

$$q_k \le q < q_{k+1}.$$

Since $q < q_{k+1}$, the contrapositive of the preceding theorem gives

$$|q_k x - p_k| \le |qx - p| = q\left|x - \frac{p}{q}\right| < q \cdot \frac{1}{2q^2} = \frac{1}{2q}.$$

Hence,

$$\left|x - \frac{p_k}{q_k}\right| < \frac{1}{2qq_k}.$$

Now assume toward a contradiction that $\frac{p}{q}$ is *not* a convergent in the continued fraction expansion for x. In particular, $\frac{p}{q} \ne \frac{p_k}{q_k}$, so $qp_k \ne pq_k$, and hence $|qp_k - pq_k|$ is a positive integer.

Since $|qp_k - pq_k| \ge 1$,

$$\frac{1}{qq_k} \le \frac{|qp_k - pq_k|}{qq_k} = \left|\frac{p_k}{q_k} - \frac{p}{q}\right| = \left|\frac{p_k}{q_k} - x + x - \frac{p}{q}\right| \le \left|\frac{p_k}{q_k} - x\right| + \left|x - \frac{p}{q}\right| < \frac{1}{2qq_k} + \frac{1}{2q^2}.$$

(The second inequality comes from the Triangle Inequality: $|a + b| \le |a| + |b|$.)

Subtracting $\frac{1}{2qq_k}$ from both sides, I get

$$\frac{1}{2qq_k} < \frac{1}{2q^2}, \quad \text{so} \quad q < q_k.$$

But I assumed $q_k \le q$, so this is a contradiction.

Therefore, $\frac{p}{q}$ is a convergent in the continued fraction expansion for x. □

Example. Show that $\frac{355}{113}$ is the best rational approximation to π by a fraction having a denominator less than 1000.

Suppose that $\frac{p}{q}$ is a fraction in lowest terms with $q < 1000$. Suppose further that

$$\left|\pi - \frac{p}{q}\right| \le \left|\pi - \frac{355}{113}\right|.$$

In other words, suppose that $\frac{p}{q}$ is at least as good an approximation as $\frac{355}{113}$.

Since $q < 1000$,

$$2q^2 < 2000000, \quad \text{so} \quad \frac{1}{2q^2} > \frac{1}{2000000} = 5 \times 10^{-7}.$$

But

$$\left|\pi - \frac{355}{113}\right| \approx 2.7 \times 10^{-7}.$$

Thus,

$$\frac{1}{2q^2} > 5 \times 10^{-7} > \left|\pi - \frac{355}{113}\right| > \left|\pi - \frac{p}{q}\right|.$$

The hypotheses of the theorem are satisfied, so $\dfrac{p}{q}$ must be a convergent in the continued fraction expansion of π. By assumption, it approximates π *at least as well* as $\dfrac{355}{113}$. But the other convergents 3, $\dfrac{22}{7}$, $\dfrac{333}{106}$ with denominators less than 1000 are *poorer* approximations to π than $\dfrac{355}{113}$. The only possibility is that $\dfrac{p}{q} = \dfrac{355}{113}$. $\square$